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Random environment on coloured trees

Mikhail Menshikov*, Dimitri Petritis†, and Stanislav Volkov‡§

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Abstract

In this paper we study a regular rooted coloured tree with random labels assigned to its edges, where the distribution of the label assigned to an edge depends on the colours of its endpoints. We obtain some new results relevant to this model and also show how our model generalizes many other probabilistic models, including random walk in random environment on trees, recursive distributional equations, and multi-type branching random walk on $\mathbb{R}$.

Keywords and phrases. Random environment on trees, random walk in random environment, branching random walks, first-passage percolation, recursive distributional equations.

AMS 2000 subject classification. Primary 60G60, 60K35; secondary 60F05, 60J80, 60K37.

1 Introduction

Random walks in random environment have been studied for a long time (see Solomon [10] for such a random walk on $\mathbb{Z}$). One of the most natural extensions of this model is to consider a random walk in random environment on a tree, see e.g. Lyons and Pemantle [7]. It turns out that the question of recurrence vs. transience of the walk is equivalent to infiniteness vs. finiteness of certain sums of random variables assigned to the edges of the tree. In [7] it is assumed that all these random variables are i.i.d., which may be a fairly restrictive condition. Indeed, in the classical setup, the probability of jump from a given vertex v through a certain edge is set to be equal to the ratio between the value assigned to this edge and the sum of the values assigned to all the edges adjacent to

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v. The assumption that these values are independent results in substantial restrictions on the possible random jump distributions assigned to vertices, in particular the symmetry of such a distribution.

Our initial motivation in writing this paper was to overcome this difficulty. Additionally, we have managed to study the situations when the distribution of value assigned to an edge depends on its direction and even on the direction of the immediately preceding edge. This has resulted in establishing two phase transitions in the model, which in turn is useful for a variety of applications - not only random walks in random environment, as outlined in Section 5.

Formally, let $b \geq 2$, and consider a b-ary regular rooted tree $T = T_b$ with root v_0 and vertex set $\mathbb{V}$ (that is a tree in which all vertices have degree $b+1$, with the exception of the root which has degree b). For any two vertices $v, u \in \mathbb{V}$ let $d(u, v)$ denote the distance between these two vertices, that is the number of edges on the shortest path connecting v and u. Let $\mathbb{V}_n$ denote the set of b^n vertices at graph-theoretical distance n from the root, and write $|v| = d(v, v_0) = n$ when $v \in \mathbb{V}_n$. If two vertices v and w are connected by an edge, we write $v \sim w$ and $\ell(v)$ will denote the sequence of vertices of the unique self-avoiding path connecting v to the root.

With the exception of the root, colour each vertex in one of b distinct colours, from left to right, such that for every fixed vertex each of its children has a different colour . For definiteness, colour the root in any of the b colours. Denote the colour of a vertex v as $c(v) \in \{1, 2, \dots, b\}$.

We are given b^2 positive-valued random variables of known joint distribution, which we denote $\bar{\xi}_{ij},\ i, j = 1, 2, \dots, b$. Now to each unoriented edge $(u, v) \equiv (v, u)$ assign a random variable ξ_{uv} such that

- for any edge (u, v), where u is *the parent* of vertex v, we have $\xi_{uv} \stackrel{\mathcal{D}}{=} \bar{\xi}_{c(u)\,c(v)}$;
- for any collection of edges $(u_1, v_1), (u_2, v_2), \dots, (u_m, v_m)$ such that u_i is the parent of v_i for all i and $u_i \neq u_j$ for all $i \neq j$, the random variables $\{\xi_{u_i v_i}\}_{i=1}^m$ are independent.

Here and throughout the paper $X \stackrel{\mathcal{D}}{=} Y$ means that random variables X and Y have the same distribution. Note that we allow dependence between sibling edges.

For any $v \in \mathbb{V}$ and any $w \in \ell(v)$ let $\xi[w, v]$ equal the product of the random variables assigned to the edges of the sub-path connecting w to v. By convention set $\xi[v, v] = 1$, and also denote $\xi[v] = \xi[v_0, v]$.

In this paper we will answer the following two questions.

Question 1. When is $\mathsf{Y} := \sum_{v \in \mathbb{V}} \xi[v]$ finite a.s.?

Question 2. Let $a > 0$. When is $\mathsf{Z} = \mathsf{Z}(a) := \text{card}\{v \in \mathbb{V} : \xi[v] > a\}$ finite a.s.?

The answers, of course, depend on the distribution of $\bar{\xi}_{ij}$'s and they are presented in Section 2, with the proofs given in Section 4, while Section 3 contains some auxiliary statements.

The study of the sum Y was a very important ingredient in the analysis carried out in [7] and it is essential for the investigation of random walks in random environment on trees, edge-reinforced random walks on trees, and some other problems. At the same time, the quantity Z is relevant to first-passage percolation and branching random walks. Some other relevant models are also mentioned in [9].

In our paper, we consider various applications as well, and they are presented in Section 5.

2 Main results

First we will introduce an alternative colouring procedure, which is equivalent to the one described above in terms of the questions we ask, yet it has some advantages. Suppose that the colouring of the tree is done in a different manner than described above. Namely, it is done recursively for $\mathbb{V}_1, \mathbb{V}_2, \dots$, as follows. Suppose that the vertices up to level $n-1$ are already coloured. Next, colour the vertices of $\mathbb{V}_n$ randomly in such a way that whenever two vertices $v \in \mathbb{V}_n$ and $u \in \mathbb{V}_n$ share a common parent, they must have different colours. The distribution of the colouring is independent of the previous levels and is uniform, that is, we assign each of the allowed $(b!)^{b^{n-1}}$ colourings with equal probability. This process can be extended to infinity, thus producing the colouring of all vertices $v \in \mathbb{V}$.

Again, assign to each edge (u, v) a random variable ζ_{uv}, such that *conditioned on the colouring of the tree*, ζ_{uv} satisfy the two conditions on ξ_{uv}'s mentioned in the previous section, and similarly compute $\zeta[w, v]$'s and $\zeta[v]$'s. Then by construction it is clear (e.g. by using coupling arguments) that for each n, the distribution of the *unordered set* $\{\zeta[v],\ v \in \mathbb{V}_n\}$ is the same as the distribution of $\{\xi[v],\ v \in \mathbb{V}_n\}$. Therefore, the answers to the questions above will be the same as in the original model. At the same time, the new model which uses randomized colouring has a significant advantage, namely

$$\text{for any two } v, w \in \mathbb{V}_n,\ \zeta[v] \stackrel{\mathcal{D}}{=} \zeta[w], \tag{2.1}$$

though $\zeta[v]$ and $\zeta[w]$ are of course dependent. Thus, from this moment on, we will only work with the new, randomly coloured model. The probability $\mathbb{P}$ and the expectation $\mathbb{E}$ below will be with respect to the measure generated by a random colouring $\mathbf{c} = \{c(v),\ v \in \mathbb{V}\}$ and a random environment $\zeta = \{\zeta_{vw},\ v, w \in \mathbb{V},\ v \sim w\}$.

For $s \in [0, \infty)$ let

$$m(s) = \begin{pmatrix} \mathbb{E}\,\left(\bar{\xi}_{11}\right)^s & \dots & \mathbb{E}\,\left(\bar{\xi}_{1b}\right)^s \\ \vdots & \ddots & \vdots \\ \mathbb{E}\,\left(\bar{\xi}_{b1}\right)^s & \dots & \mathbb{E}\,\left(\bar{\xi}_{bb}\right)^s \end{pmatrix}$$

and let $\rho(s)$ be the largest eigenvalue of $m(s)$, which is positive by the Perron-Frobenius Theorem, since all the elements of the matrix are strictly positive and hence it is irreducible.

We will need the following regularity conditions. Let

$$\mathbb{D} = \left\{ s \in \mathbb{R} : \ \mathbb{E}\, \bar{\xi}_{ij}^{s} < \infty \ \forall i, j \in \{1, 2, \ldots, b\} \right\}$$

and suppose that

$$\begin{array}{rcl} [0,1] & \subseteq & \mathbb{D}, \\ 0 & \in & \mathsf{Int}(\mathbb{D}), \\ \mathbb{E}\, |\log \bar{\xi}_{ij}| & < & \infty \ \forall i, j \in \{1, 2, \ldots, b\}, \\ \mathbb{E}\, |\bar{\xi}_{ij} \log \bar{\xi}_{ij}| & < & \infty \ \forall i, j \in \{1, 2, \ldots, b\}. \end{array} \tag{2.2}$$

Now we are ready to present our main results.

Theorem 1 *Let* $\mathsf{Y} = \sum_{v \in \mathbb{V}} \zeta[v]$ *and* $\lambda_1 = \inf_{s \in [0,1]} \rho(s)$.

> *(a) If* $\lambda_1 < 1$ *then* $\mathsf{Y} < \infty$ *a.s.*
>
> *(b) If* $\lambda_1 > 1$ *and the conditions (2.2) in the next section are fulfilled, then* $\mathsf{Y} = \infty$ *a.s.*

Theorem 2 *Let* $x > 0$, $\mathsf{Z}(x) = card\{v \in \mathbb{V} : \zeta[v] > x\}$ *and* $\lambda = \inf_{s \in [0,\infty)} \rho(s)$. *Additionally suppose (2.2) are fulfilled.*

> *(a) If* $\lambda < 1$ *then* $\mathsf{Z}(x) < \infty$ *a.s.*
>
> *(b) If* $\lambda > 1$, *then* $\mathsf{Z}(x) = \infty$ *a.s.*

Note that we do not attempt to analyze here the situation in the critical case $\lambda = 1$ ($\lambda_1 = 1$ resp.) The reason is that unlike the one-dimensional situation, the analysis becomes much more difficult here and we could not find any reasonable and interesting conditions which would ensure infiniteness of $\mathsf{Z}(x)$ or Y.

3 Large deviations results

Let $v \in V_n$ and suppose that $\ell(v) = \{v_0, v_1, \ldots, v_{n-1}, v_n = v\}$. Then the random variables $c(v_i)$, $i = 1, 2, \ldots, n$ are i.i.d. random variables with uniform distribution on the set $\{1, 2, \ldots, b\}$.

Lemma 1 *Let* $S_n = \sum_{i=1}^{n} \log(\zeta_{v_{i-1} v_i})$ *and*

$$k_n(s) = \left(\mathbb{E}\left(e^{s S_n} \right) \right)^{1/n} = \left(\mathbb{E} \prod_{i=1}^{n} \zeta_{v_{i-1} v_i}^{s} \right)^{1/n}.$$

Suppose (2.2) are fulfilled. Then

(a) $k(s) = \lim k_n(s) \in [0, \infty]$ *exists for all* s*;*

(b) $\Lambda(s) = \log \rho(s) - \log b = \log k(s) \in (-\infty, +\infty]$ *is convex;*

(c) the rate function $\Lambda^*(z) = \sup_{s \in \mathbb{D}}(sz - \Lambda(s))$, $z \in \mathbb{R}$ *is convex, lower semi-continuous and differentiable in* $\textsc{Int}(\mathbb{D})$*. Moreover*

$$\Lambda^*(z) = \begin{cases} s_0(z)z - \Lambda(s_0(z)) & \textit{if } z \geq \Lambda'(0) \\ 0 & \textit{if } z \leq \Lambda'(0), \end{cases}$$

where $s_0(z)$ *is the solution of equation* $z - \Lambda'(s) = 0$*;*

(d) for all $a > 0$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \geq \log a\right) = -\Lambda^*(\log a).$$

Remark 1 *The statement of the Lemma holds simultaneously for all possible colourings of the root.*

Proof.

First, for any $s_1, s_2 \in \mathbb{D}$, with $s_1 < s_2$, the segment $[s_1, s_2]$ belongs to $\mathbb{D}$, hence for any $\alpha \in (0,1)$, we have $k_n^n(\alpha s_1 + (1-\alpha)s_2) = \mathbb{E}(\exp(\alpha s_1 S_N))\exp((1-\alpha)s_2 S_N)) \leq [\mathbb{E}(\exp(s_1 S_N))]^\alpha [\mathbb{E}(\exp(s_2 S_N))]^{1-\alpha}$ from where logarithmic convexity of k_n follows.

Suppose $c(v_0) = \alpha$. Then $k_n(s) = \frac{1}{b}\left(e_\alpha^T m(s)^n e\right)^{1/n}$ where

$$e_\alpha = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \alpha\text{-th position}, \quad e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{3.3}$$

Now $m(s) = \rho_1(s)P_1 + \rho_2(s)P_2 + \ldots$ where (ρ_i) are the eigenvalues of $m(s)$, ordered so that $\rho_1 > |\rho_2| \geq |\rho_3| \geq \ldots$, and P_i denotes the projection on the $i-$th eigenspace corresponding to the $i-$th eigenvalue ρ_i. Notice that $\rho \equiv \rho_1 > 0$, the image space of P_1 is $1-$dimensional and since $m(s)_{ij} > 0$ for all i, j, we have

$$(P_1 e)_i > 0 \text{ for all } i \quad \text{and} \quad |\rho_2| < \rho_1.$$

Hence $k(s) = \frac{1}{b}\rho(s) \in (0, \infty]$ for all s and is log-convex, as limit of a log-convex function is log-convex.

Finally (d) follows from Gärtner-Ellis theorem (see, for instance, Lemma V.4 p. 53 in den Hollander [4]), under conditions (2.2). ∎

Note that we can rewrite Λ^* as

$$\begin{aligned}\Lambda^*(z) &= \sup_{s\ge 0}[sz - \log(\rho(s)/b)] \\ &= -\log \inf_{s\ge 0} \frac{\rho(s)e^{-sz}}{b}.\end{aligned} \tag{3.4}$$

Recall that

$$\begin{aligned}\lambda &= \inf_{s\ge 0} \rho(s), \\ \lambda_1 &= \inf_{s\in[0,1]} \rho(s) \ge \lambda.\end{aligned}$$

Corollary 1 *For any $\tilde{\lambda} < \lambda_1$ and $\alpha \in \{1, \ldots, b\}$ there exists a $y \in (0, 1]$ and a positive integer n such that for any $v, u \in \mathbb{V}$ such that $u \in \ell(v)$, $c(u) = \alpha$, and $d(u, v) = n$*

$$\mathbb{P}\left(\zeta[u, v] > y^n\right) \ge \frac{\tilde{\lambda}}{(by)^n}.$$

Proof.

Lemma 1 yields that for any small $\varepsilon > 0$ and all $y > 0$ there is an $n_0 = n_0(\varepsilon, y)$ such that for every $n \ge n_0$

$$\begin{aligned}e^{n\varepsilon}\,\mathbb{P}(S_n/n \ge \log y) &\ge \exp\{-n\Lambda^*(\log y)\} = \exp\left\{n \inf_{s\ge 0}[\log(\rho(s)/b) - s\log y]\right\} \\ &= \left(\inf_{s\ge 0} \frac{\rho(s)y^{-s}}{b}\right)^n = \frac{1}{(yb)^n}\left(\inf_{s\ge 0} \rho(s)y^{1-s}\right)^n\end{aligned}$$

whence for any $v, u \in \mathbb{V}$ such that $u \in \ell(v)$ and $|v| = |u| + n$

$$\mathbb{P}(\zeta[u, v] \ge y^n) \ge \left[\frac{e^{-\varepsilon}}{yb} \times \inf_{s\ge 0} \rho(s)y^{1-s}\right]^n$$

Now since $\log \rho(s)$ is convex, it follows from the proof of the Lemma on p. 129 in Lyons and Pemantle [7] that

$$\max_{0<y\le 1} \inf_{s\ge 0} \rho(s)y^{1-s} = \min_{0\le s\le 1} \rho(s) = \lambda_1.$$

Consequently, by choosing $y \in (0, 1]$ at the point where this maximum is achieved, and $\varepsilon > 0$ very small, we ensure in fact that for all large n

$$\mathbb{P}(\zeta[u, v] \ge y^n) \ge \frac{\tilde{\lambda}}{(yb)^n}. \tag{3.5}$$

■

4 Proofs of the main theorems

Proof of Theorem 1.

(a) Suppose that $\lambda_1 < 1$. Then we can fix an $s \in (0,1)$ such that $\rho(s) < 1$. Suppose that the root has color α. Then

$$\sum_{v \in \mathbb{V}_n} \zeta^s[v] = \sum_{\ell(v)=(v_0,\ldots,v_n):\ v=v_n \in \mathbb{V}_n} \zeta^s_{v_0 v_1} \zeta^s_{v_1 v_2} \cdots \zeta^s_{v_{n-1} v_n}$$

and hence by construction of the colouring of the tree, we have

$$\begin{aligned} \mathbb{E}\left(\sum_{v \in \mathbb{V}_n} \zeta^s[v]\right) &= \sum_{\ell(v)=(v_0,\ldots,v_n):\ v=v_n \in \mathbb{V}_n} m_{\alpha c(v_1)}(s) m_{c(v_1)c(v_2)}(s) \ldots m_{c(v_{n-1})c(v_n)}(s) \\ &= e_\alpha^T m^n(s) e \end{aligned}$$

where e_α and e are defined in (3.3). Now, since $\rho(s) < 1$, $\sum_{n=1}^{\infty} m^n(s) < \infty$, therefore by Fubini's theorem $\mathbb{E}\left(\sum_v \zeta^s[v]\right) < \infty$ and hence $\sum_v \zeta^s[v]$ is finite a.s. This implies that $\zeta[v] \geq 1$ for *only finitely* many v's, and therefore there exists an N such that whenever $v \notin \mathbb{V}_1 \cup \cdots \cup \mathbb{V}_N$ it follows $\zeta[v] < 1$, hence $\zeta^s[v] > \zeta[v]$. Consequently,

$$\begin{aligned} \mathsf{Y} &= \sum_{i=1}^{N} \left(\sum_{v \in \mathbb{V}_i} \zeta[v]\right) + \sum_{i=N+1}^{\infty} \left(\sum_{v \in \mathbb{V}_i} \zeta[v]\right) \\ &< \sum_{i=1}^{N} \left(\sum_{v \in \mathbb{V}_i} \zeta[v]\right) + \sum_{i=N+1}^{\infty} \left(\sum_{v \in \mathbb{V}_i} \zeta^s[v]\right) < \infty. \end{aligned}$$

where the last inequality follows from the fact that $\sum_v \zeta^s[v]$ is finite a.s.

(b) Since $\lambda_1 > 1$, by Corollary 1, there are an $\varepsilon > 0$, $0 < y \leq 1$ and an n such that for any $v, u \in \mathbb{V}$ satisfying $u \in \ell(v)$ and $|v| = |u| + n$

$$\mathbb{P}\left(L[u,v]\right) \geq \frac{1+\varepsilon}{(by)^n} =: q,$$

where

$$L[u,v] := \{\zeta[u,v] \geq y^n\}.$$

Let us construct an embedded branching process, with members of generation j denoted M_j, as follows. The root of the tree v_0 is the sole member of generation 0, that is $M_0 = \{v_0\}$. For $j \geq 1$, let

$$M_j = \{u \in \mathbb{V}_{jn} : \exists\, w \in M_{j-1} \quad \begin{array}{cl} \text{such that} & \mathbb{V}_{(j-1)n} \cap \ell(u) = \{w\} \\ \text{and} & L[w,u] \text{ occurs}\} \end{array} \tag{4.6}$$

The process $|M_j|$ can be minorised by an *independent* branching process with uniformly bounded number of descendants whose average is equal to

$$\mu := b^n \times q = (1+\varepsilon) y^{-n} > 1,$$

which is a supercritical process surviving with a positive probability, say $p_S > 0$; moreover

$$\left\{ \lim_{j\to\infty} \frac{|M_j|}{\mu^j} > 0 \right\} = \{\text{the process survives}\} \quad \text{a.s.}$$

by Kesten-Stigum theorem, see e.g. [1], p. 192. This, in turn, implies that there is a positive $\delta > 0$ such that with probability at least $p_S/2 > 0$ for all j large enough we have $|M_j| \geq \delta\mu^j$. Consequently, since for each $v \in M_j$ we have $\zeta[v] \geq (y^n)^j$,

$$\mathsf{Y} \geq \sum_{v\in M_j} \zeta[v] \geq \delta\mu^j (y^n)^j = \delta(1+\varepsilon)^j \to \infty \text{ as } j \to \infty$$

with positive probability. Now, the set $\{\mathsf{Y} = \infty\}$ is a tail event and the random variables at different generations are independent, hence its probability satisfies the $0-1$ law and we obtain the required result. ■

Proof of Theorem 2.

(a) Recall that for any $v \in \mathbb{V}_n$ the quantity $p_n = \mathbb{P}(\zeta[v] > x)$ does not depend on v, and observe that

$$\mathbb{E}\,\mathsf{Z}(x) = \sum_{n=1}^{\infty} b^n p_n.$$

Since $\lambda < 1$, from (3.4) we have that for a small $z < 0$ and a very small $\varepsilon > 0$

$$-\Lambda^*(z) < \log \frac{1-2\varepsilon}{b}.$$

Set $y = e^z < 1$ and apply Lemma 1 (d) to obtain that for all large n

$$\frac{1}{n} \log \mathbb{P}(\zeta[v] \geq y^n) \leq \log \frac{1-\varepsilon}{b}.$$

This yields

$$b^n \mathbb{P}(\zeta[v] \geq y^n) \leq (1-\varepsilon)^n,$$

and since $x > 0$ and $y < 1$ implies $p_n = \mathbb{P}(\xi[v] > x) \leq \mathbb{P}(\xi[v] \geq y^n)$ for large n, we have $\mathbb{E}\,\mathsf{Z}(x) < \infty$, and so $\mathsf{Z}(x) < \infty$ a.s.

(b) Now since $\lambda > 1$, from (3.4) we have that for a small $z > 0$ and a very small $\varepsilon > 0$,

$$-\Lambda^*(z) > \log \frac{1+2\varepsilon}{b}.$$

As before, we set $y = e^z > 1$, and apply Lemma 1 (d), to obtain that there is an n, which we fix from now on, such that

$$\frac{1}{n} \log \mathbb{P}(\zeta[v] \geq y^n) \geq \log \frac{1+\varepsilon}{b} \quad \Longrightarrow \quad b^n \mathbb{P}(\zeta[v] \geq y^n) \geq (1+\varepsilon)^n. \qquad (4.7)$$

Next we construct a branching process, that is almost identical to the one constructed in the proof of Theorem 1. Again, provided that $u \in \ell(v)$ and $|v| - |u| = n$, we introduce the event

$$L[u, v] := \{\zeta[u, v] \geq y^n\}, \tag{4.8}$$

whose probability is at least $(1+\varepsilon)^n/b^n$ according to (4.7). Let the root of the tree v_0 be the unique member of generation 0, that is $M_0 = \{v_0\}$. Similarly to the previous proof, for $j \geq 1$ let M_j be defined by (4.6). Then the process $|M_j|$ can again be minorised by a supercritical independent branching process, with average number of descendants equal to $\mu := (1+\varepsilon)^n > 1$, which survives with a positive probability $p_S > 0$. On the event $\sum_{j=1}^{\infty} |M_j| = \infty$ of survival, for any $x > 0$ there exists $j_0 = j_0(x)$ such that for all $j \geq j_0$, we have $v \in M_j$ implying $\zeta[v] \geq y^{nj} > x$. Consequently,

$$\mathbb{P}(\mathsf{Z}(x) = \infty \text{ for all } x > 0) \geq p_S > 0.$$

Taking into account the fact that the event $\{\mathsf{Z}(x) = \infty \text{ for all } x > 0\}$ is a tail event and variables at different generations are independent, we conclude that for any $x > 0$

$$\mathbb{P}(\mathsf{Z}(x) = \infty) = 1.$$

■

5 Applications

Here we show how Theorems 1 and 2 can be applied to obtain some of the already known facts as well as to establish new results in various applications of probability theory. Throughout this section, we will assume that the regularity conditions (2.2) are satisfied.

5.1 Random walk in random environment

Let u be a vertex of the coloured tree T. For every $v \sim u$ define $p_{uv} \in (0, 1)$ such that $\sum_{v:\ v\sim u} p_{uv} = 1$. For definiteness, denote the parent of u as u^*, and the children of u as $u^1, u^2, \dots, u^b$ (also, when u is the root v_0 of the tree, set $u^* \equiv u$). Now suppose that for each u

$$\mathbf{p}(u) = (p_{uu^*}, p_{uu^1}, p_{uu^2} \dots, p_{uu^b}) \in (0, 1)^{b+1}$$

is a $(b+1)$-dimensional random variable. Obviously, the set of the components of $\mathbf{p}(u)$ is *dependent*, since they have to sum up to 1.

Suppose that the distribution of $\mathbf{p}(u)$ depends only on the colour $c(u)$ of the vertex u. Additionally suppose that the random variables $\{\mathbf{p}(u), u \in \mathbb{V}\}$ are independent.

Now define a random walk $X(k)$ in a random environment on the coloured tree T by letting $X(0) = v_0$ and

$$\mathbb{P}(X(k+1) = v \mid X(k) = u) = \begin{cases} p_{uv} & \text{if } u \sim v, \\ 0 & \text{otherwise,} \end{cases}$$

where we set $v_0 \sim v_0^* = v_0$. This model is similar to that of Lyons and Pemantle [7]; however, we do not require as much independence or symmetry for the distribution of the jumps to children, as required in [7]. Additionally, we also allow jump distributions to depend on the type of the vertex. On the other hand, in [7] more general trees have been considered, while we restrict ourselves only to regular trees.

We want to establish when the walk in the random environment is transient (resp., recurrent). For $i = 1, 2, \ldots, b$ let

$$(\bar{\xi}_{i1}, \bar{\xi}_{i2}, \ldots, \bar{\xi}_{ib}) \stackrel{\mathcal{D}}{=} \left(\frac{p_{uu^1}}{p_{uu^*}}, \frac{p_{uu^2}}{p_{uu^*}}, \ldots, \frac{p_{uu^b}}{p_{uu^*}}, \right)$$

whenever $c(u) = i$. Also let $m(s)$, $\rho(s)$, and λ_1 be the same as defined in Section 2.

Proposition 1 *The random walk in random environment described above is a.s. positive recurrent when $\lambda_1 < 1$, and a.s. transient when $\lambda_1 > 1$.*

Proof.

We will use the standard electric network representation of the random walk by replacing each edge of the colored tree T with a resistor, such that their conductances satisfy the following formula:

$$\frac{C_{uu^*}}{C_{uu^i}} = \frac{p_{uu^*}}{p_{uu^i}}$$

where again $u^1, \ldots, u^b$ are the children of vertex u and u^* is its parent (see [5]). These equations are satisfied when for any $u \in \mathbb{V}_n$, $n \geq 1$, with $\ell(u) = \{u_0 \equiv v_0, u_1, u_2, \ldots, u_{n-1}, u_n \equiv u\}$ we have[1]

$$C_{u_{n-1}u_n} = \prod_{i=0}^{n-1} \frac{p_{u_i u_{i+1}}}{p_{u_i u_{i-1}}}$$

Now to each edge (u^*, u) where u^* is the parent of u, assign a random variable with distribution $\bar{\xi}_{c(u^*)c(u)}$. Then $C_{u_{n-1}u_n}$ is equal to the product of the random variables assigned to edges of the path connecting v_0 to $u_n = u$. Theorem 1 implies that whenever $\lambda_1 < 1$, $\mathsf{Y} = C := \sum_{x,y} C_{x,y} < \infty$ a.s., and then there exists a stationary probability measure π such that $\pi_x = C_x / C$ where

$$C_x = \sum_{y:\ y \sim x} C_{xy}.$$

[1] When $u_i = v_0$, we need to set $p_{u_i u_{i-1}} = p_{v_0 v_0}$.

Therefore, the random walk is positive recurrent.

The reverse statement (about transience for $\lambda_1 > 1$) follows from a slight modification of the proof of part (i) of Theorem 1 of [7], effectively using the estimate (3.5), since transience is equivalent to establishing finiteness of the effective resistance R_{eff}. ■

Example. Consider a random walk in a random environment on a coloured binary tree ($b = 2$). Suppose that from a vertex of type 1, the walk always goes down with probability $\frac{1}{2}$ and up with probabilities $\frac{1}{4}$ to either of its children. From a vertex of type 2, the walk goes up and right with probability $\frac{1}{4}$, down with probability $\frac{3}{4}\eta_v$ and up and left with probability $\frac{3}{4}(1-\eta_v)$, where η_v are i.i.d. random variables distributed uniformly on $[h, 1]$, $h \in (0, 1)$. Then

$$m(s) = \begin{pmatrix} 2^{-s} & 2^{-s} \\ \mathbb{E}\left(\frac{1-\eta}{\eta}\right)^s & \mathbb{E}\left(\frac{1}{3\eta}\right)^s \end{pmatrix}.$$

It is easy to verify that if $\rho(s)$ is the largest eigenvalue of $m(s)$, then $\lambda_1 = \inf_{s\in[0,1]} \rho(s)$ is smaller than 1 whenever $h > h_{cr} = 0.417\ldots$, and thus the walk is positive recurrent for almost every environment when $h > h_{cr}$, and transient for almost every environment when $h < h_{cr}$.

5.2 Recursive distributional equations

It turns out that our construction on randomly coloured trees may be used to answer the question about the existence of solutions of certain functional equations, described below.

Let Ξ be the $b \times b$ matrix of random variables $\bar{\xi}_{ij}$, $i, j = 1, 2, \ldots, b$. We want to find a b-dimensional random vector $Y = (Y_1, \ldots, Y_b)^T$, independent of Ξ, such that

$$1 + \sum_{j=1}^{b} \bar{\xi}_{ij} Y_j \stackrel{\mathcal{D}}{=} Y_i \quad \text{for} \quad i = 1, 2, \ldots, b \tag{5.9}$$

which can be expressed in a vector form as

$$e + \Xi Y \stackrel{\mathcal{D}}{=} Y$$

where e has been defined by (3.3). Equation (5.9) is a special case of a more general recursive distributional equation, which have been widely studied; for example see [8] where one can find sufficient conditions for the existence of its solution (Theorem 4.1). At the same time, for equation (5.9) we obtain essentially a *criterion* for this existence.

Note that it will be essential that the tree is coloured; otherwise we would have been able to solve (5.9) only in a one-dimensional case $b = 1$.

Proposition 2 *Let* Y *be the quantity defined in Question 1 of Section 1. Then equation (5.9) has a solution if and only if* $\mathsf{Y} < \infty$ *a.s.*

Proof.

First, suppose $\mathsf{Y} < \infty$ a.s. For each $i = 1, \ldots, b$ let $Y_i = \sum_{v \in \mathbb{V}} \zeta[v]$ when $c(v_0) = i$, and suppose that different Y_i's are constructed using independent random variables. By assumption, each $Y_i < \infty$ a.s. Now it is easy to see that $1 + \sum_{i=1}^{b} \bar{\xi}_{ij} Y_j$ indeed has the distribution of Y_i hence $Y = (Y_1, \ldots, Y_b)^T$ is a solution of (5.9).

Secondly, suppose $\mathsf{Y} = \infty$ a.s., and suppose that there is a solution $\hat{Y} = (Y_1, Y_2, \ldots, Y_n)^T$ of equation (5.9). Construct a $b-$ary tree with $c(v_0) = 1$ and assigned random variables $\bar{\xi}$'s as described in the introduction. Also for each n and each $v \in \mathbb{V}_n$ let $Q(v)$ be an independent random variable with the distribution of $Y_{c(v)}$, and denote

$$\begin{aligned} \mathsf{Y}_1^{(<n)} &= \sum_{v \in \mathbb{V}_0 \cup \cdots \cup \mathbb{V}_{n-1}} \zeta[v], \\ \tilde{\mathsf{Y}}_1^{(<n)} &= \sum_{v \in \mathbb{V}_0 \cup \cdots \cup \mathbb{V}_{n-1}} \zeta[v] + \sum_{v \in V_n} \zeta[v] Q(v) \end{aligned}$$

for $n = 1, 2, \ldots$. Since $\hat{Y}$ is a solution to the problem, it follows by induction on n that $\tilde{\mathsf{Y}}_1^{(<n)}$ must have the distribution of Y_1 and this is true for *all* n. Now observe that $\mathsf{Y}_1^{(<n)} \le \tilde{\mathsf{Y}}_1^{(<n)}$. At the same time, $\lim_{n\to\infty} \mathsf{Y}_1^{(<n)} = \mathsf{Y} = \infty$ a.s. by assumption. Hence $\tilde{\mathsf{Y}}_1^{(<n)}$ which equals to Y_1 in distribution, is larger than a random variable equal to ∞ a.s., which is impossible. ■

Let λ_1 be the same as defined in Section 2. Then Theorem 1 yields the following

Corollary 2 *If* $\lambda_1 < 1$ *then there is a solution to equation (5.9). In contrast, if* $\lambda_1 > 1$ *then equation (5.9) has no solution.*

5.3 First-passage percolation

Here we show how our techniques can extend the results of the first passage percolation theory to the situation where one allows *negative passage times.* For each edge (u, v) of the coloured tree T, where u is the parent of v, let τ_{uv} denote the passage time from vertex u to vertex v. Allow these times to be also negative, for example, indicating a "speed up" of a walker. Suppose for simplicity that τ_{uv}'s are all independent, while their distribution depends on the colour of the endpoints, thus being one of the b^2 possible types. Let

$$R(t) = \{u \in \mathbb{V} : \sum_{(v,w) \in \ell(u)} \tau_{vw} \le t\}$$

be the set of the vertices of the tree, reachable in time t. The primary question is whether $R(t)$ is finite, since it can be easily infinite due to the negative passage times.

To answer this, for all u and v such that $u \sim v$ and u is the parent of v set $\bar{\xi}_{c(u)c(v)} \stackrel{\mathcal{D}}{=} e^{-\tau_{uv}}$. The following statement is straightforward.

Proposition 3 *$R(t)$ is finite a.s. if and only if the quantity $\mathsf{Z}(e^{-t})$ defined in Question 2 of Section 1 is finite a.s.*

5.4 Multi-type branching random walks on $\mathbb{R}^1$

The literature on the branching random walks is fairly extensive, and a similar model to the one which follows was considered for example in [2], although somewhat different questions were investigated in that paper.

Suppose that we are given b^2 positive-valued random variables η_{ij}, $i, j = 1, 2, \ldots, b$ with known joint distribution. Consider a process on $\mathbb{R}$ which starts with a single particle of type $i \in \{1, 2, \ldots, b\}$ located at point $X^{(0)} = 0 \in \mathbb{R}$. The particle splits into b new particles, one of each type $1, 2, \ldots, b$. If the positions of the new particles of types $1, 2, \ldots, b$ are denoted $X_1^{(1)}, X_2^{(1)}, \ldots, X_b^{(1)}$ respectively, then the distribution of jumps $X_j^{(1)} - X^{(0)}$ are independent for different j's and have the distribution of η_{ij}. After this, each of the new particles behaves in the same way as the original particle, so by time $t \in \{1, 2, \ldots\}$ we will have exactly b^t particles $X_1^{(t)}, \ldots, X_{b^t}^{(t)}$ located somewhere on $\mathbb{R}$.

Set $\bar{\xi}_{ij} = \exp(-\eta_{ij})$, $i, j = 1, \ldots, b$. Then the following statement is obvious.

Proposition 4 *Suppose that $\mathsf{Z}(1)$, as defined in Question 2 of Section 1, is finite a.s. Then all the particles will be eventually on the positive semi-axis a.s., that is*

$$\mathbb{P}\left(\exists N:\ \forall t \geq N \min_{i \in \{1, 2, \ldots, b^t\}} X_i^{(t)} \geq 0\right) = 1.$$

In fact, we can strengthen this result. Let

$$\mu_t = \min_{i=1,\ldots,b^t} X_i^{(t)}$$

be the minimum displacement of our multi-type branching random walk (see e.g. [3] and references therein). As before, set $\bar{\xi}_{ij} = \exp(-\eta_{ij})$, $i, j = 1, \ldots, b$ and let $\rho(s)$ be the largest eigenvalue of

$$m(s) = \begin{pmatrix} \mathbb{E}\, e^{-s\eta_{11}} & \ldots & \mathbb{E}\, e^{-s\eta_{1b}} \\ \vdots & \ddots & \vdots \\ \mathbb{E}\, e^{-s\eta_{b1}} & \ldots & \mathbb{E}\, e^{-s\eta_{bb}} \end{pmatrix}.$$

For $x \in \mathbb{R}$ let

$$\lambda^{(x)} = \inf_{s \geq 0} e^{sx} \rho(s), \tag{5.10}$$

and observe that $\lambda^{(x)}$ is non-decreasing in x. Note that if the joint distribution of η_{ij}'s is not degenerate, then $\rho(s)$ is strictly log-convex in s and therefore there exists a unique x_0 such that $\lambda^{(x_0)} = 1$.

Proposition 5 *Under the non-degeneracy condition above,*

$$\lim_{t\to\infty} \frac{\mu_t}{t} = x_0 \quad a.s.,$$

where x_0 is the unique solution of the equation $\lambda^{(x_0)} = 1$.

Proof. For each $x \in \mathbb{R}$ we can define a new multi-type branching random walk with the step sizes equal to $\eta_{ij}^{(x)} = \eta_{ij} - x$ for every i and j. Then we can naturally couple the new walk $X_k^{(t;\,x)}$ with the original one by setting $X_k^{(t;\,x)} = X_k^{(t)} - tx$. Observe that the largest eigenvalue $\rho^{(x)}(s)$ of the matrix $m^{(x)}(s)$ for this modified walk, whose entries are $\left(\mathbb{E}\, e^{-s(\eta_{ij}-x)}\right)_{i,j=1}^{b}$, equals $e^{sx}\rho(s)$, hence the value of λ needed for Theorem 2 is given by (5.10).

Suppose $x < x_0$. Then $\lambda^{(x)} < 1$ whence by Theorem 2 $Z(1) < \infty$ a.s. and by Proposition 4

$$\mathbb{P}(\mu_t - tx \geq 0 \text{ for all sufficiently large } t) = 1. \tag{5.11}$$

To prove the complementary statement, we need to improve slightly the proof of part (b) of Theorem 2. First, choose $x > x_0$ yielding $\lambda^{(x)} > 1$, and replace the event (4.8) by

$$\tilde{L}[u,v] := \{\zeta[u,v] \geq y^n, \text{ and } \zeta[u,w] > \nu \text{ for all } w \in \ell(v) \text{ with } |w| > |u|\}.$$

One can choose the constants $\nu > 0$ and $\varepsilon > 0$ so small, that still $\mathbb{P}(\tilde{L}[u,v]) > (1+\varepsilon)^n/b^n$. Then we can construct sets M_j defined by (4.6) with $L[u,v]$ replaced by $\tilde{L}[u,v]$.

Let j_0 be so large that $\nu\, y^{nj_0} > 1$ (recall that $y > 1$). On the event of survival of the process $|M_j|$, for each j we have $M_j \neq \emptyset$ and $M_{j+1} \neq \emptyset$, hence there are a $u \in M_j \subseteq \mathbb{V}_{nj}$ and a $v \in M_{j+1} \subseteq \mathbb{V}_{n(j+1)}$ such that $u \in \ell(v)$ and $\tilde{L}[u,v]$ occurs. Consequently, for every $t \geq nj_0$ such that $nj \leq t < n(j+1)$ there is a $w \in \mathbb{V}_t$ such that $w \in \ell(v)$, whence $\zeta[w] = \zeta[u,w]\,\zeta[u] \geq \nu y^{nj} > 1$. On the other hand $\zeta[w] > 1$ implies $\mu_t - tx < 0$. Therefore, we have proved that the event

$$\{\mu_t - tx < 0 \text{ for all sufficiently large } t\} \tag{5.12}$$

has positive probability, since the branching process minorising $|M_j|$ is supercritical. However, the event (5.12) is a tail event, so it must have probability 1. Together with (5.11) this finishes the proof of Proposition 5. ■

5.5 Number theory: $5x+1$ Collatz-type problem

Fix an odd positive integer q and define the following map:

$$T_q : x \to \begin{cases} x/2, & \text{if } x \text{ is even,} \\ qx+1, & \text{if } x \text{ is odd.} \end{cases}$$

The famous and yet unresolved Collatz problem (see e.g. [6] for hundreds of references on papers and short description of their content) states that if one sequentially applies mapping T_3 to any positive integer, then eventually it will arrive to the cycle $1 \to 4 \to 2 \to 1$. On the other hand, a similar mapping T_5 is conjectured to "explode", i.e., for most positive integers $\lim_{n\to\infty} T_5^{(n)}(x) = \infty$ (Ya.G. Sinai, personal communications).

Another conjecture made in [11] states that the density of those numbers $x \in \mathbb{Z}_+$ for which $\lim_{n\to\infty} T_5^{(n)}(x) < \infty$ has a "Hausdorff dimension" approximately 0.68, and this conjecture was made based on a construction of a probabilistic "equivalent" of mapping T_5, leading to a special case of the model studied for answering Question 2. For more details, see [11].

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